On the existence of longitudinal or flexural waves in rods at nonlinear higher harmonics

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Abstract

This article theoretically studies the conditions for existence of longitudinal or flexural waves in nonlinear, isotropic rods. It has been known that the existence of guided waves at nonlinearity induced double harmonics is subject to constraints which arise from the potential of power flux transfer from the primary generating mode to the generated higher order modes. The knowledge about the behavior of waves in rods at harmonics higher than double is still limited. This gap was here addressed by the method of perturbation coupled with wavemode orthogonality and forced response. This reduces the nonlinear problem to a forced linear problem which is subsequently investigated to formulate an angular order-based constraint as the condition of existence/nonexistence of nonlinearity-driven higher harmonics of longitudinal and flexural waves in rods.

Key words: Nonlinear guided waves, Higher harmonics, Rod waves, Perturbation

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1 Introduction

In an earlier article [1], the authors conducted theoretical studies on the symmetry characteristics of Rayleigh-Lamb guided waves in nonlinear, isotropic plates. Considering weak nonlinearity, the problem was reduced to a set of forced linear problems following the earlier works by de Lima and Hamilton [2] and Auld [3]. The problem was developed to formulate an energy level constraint as the defining factor for the absence of antisymmetric Lamb waves at any order of *even* higher harmonics. The energy constraint also indicated the potential presence of both antisymmetric and symmetric Lamb waves at any order of *odd* harmonics.

The purpose of the present article is to extend the study of plate waves of Ref. [1] to the case of rod waves. Assuming weak nonlinearity, the strain energy is expressed as a summation series of powers of strain components. The analysis is similar to that by de Lima and Hamilton [4] which solved the problem for the first order nonlinearity. This problem is here extended by breaking several orders of nonlinearity to several forced problems, each corresponding to a single order of higher harmonic. It was found that the angular order of the primary generating mode determines which modes have the potential of being generated at either even or odd higher harmonic.

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2 Statement of the nonlinear problem

The equation of motion for nonlinear elasticity in a stress free rod is given by (Fig. 1)

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) + \mathbf{f} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{1}$$

with the stress free boundary condition:

$$[\mathbf{S^L}(\mathbf{u}) - \bar{\mathbf{S}}(\mathbf{u})] \, . \, \mathbf{n_r} = \mathbf{0} \quad on \quad \Gamma \tag{2}$$

where $\mathbf{u}$ is the particle displacement, λ and μ are the Lamé constants, ρ_0 is the initial density of the body, $\mathbf{f}$ is the body force, $\mathbf{n_r}$ is the unit vector normal to the surface of the waveguide Γ, $\mathbf{S^L}$ and $\bar{\mathbf{S}}$ are the linear and nonlinear parts of the second Piola-Kirchoff stress tensor, respectively.

Energy is written in Murnaghan potentials [5]:

$$E = \phi_2 + \phi_3 + \phi_4 ... \tag{3}$$

where ϕ_n corresponds to the set of terms in the energy expression which are of degree n in strain multiples. For example, ϕ_2 corresponds to the linear regime whereas ϕ_3 corresponds to 1^{st} order nonlinearity. The energy expression is ultimately a function of the three invariants of the 2^{nd} order strain tensor, I_1, I_2, and I_3 [6]:

$$I_1 = \varepsilon_{ij} g^{ij} \tag{4}$$

$$I_2 = \varepsilon_{im} \varepsilon_{nk} g^{ik} g^{nm} \tag{5}$$

$$I_3 = \varepsilon_{pm} \varepsilon_{in} \varepsilon_{kq} g^{im} g^{pq} g^{kn} \tag{6}$$

where g^{ij} are components of the metric tensor for the coordinate system under consideration.

Strain can be expressed in terms of covariant differentials of displacements:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i} + u_{k;i} u^k_{;j}) \tag{7}$$

The covariant differentials are expressed as:

$$u_{i;j} = u_{i,j} - u_k \Gamma^k_{ij} \tag{8}$$

and

$$u^i_{;j} = u^i_{,j} + u^k \Gamma^i_{kj} \tag{9}$$

where Γ^i_{jk} are the Christoffel symbols. Stress is expressed by:

$$\sigma^{ij} = \frac{\partial E}{\partial \varepsilon_{ij}} \tag{10}$$

and the body force vector is

$$f^i = \sigma^{ij}_{;j} \tag{11}$$

3 Solution to the nonlinear problem

3.1 Forced solution to guided waves

Following Auld [3] and de Lima and Hamilton [4], the solution to a solid rod under arbitrary surface and body forces can be written as a linear combination of the existing guided wavemodes (Fig. 1):

$$\mathbf{v}(\mathbf{r}, z, t) = \frac{1}{2}\sum_{m=1}^{\infty} A_m(z)\mathbf{v}_m(\mathbf{r})e^{-i\omega t} \quad + \quad c.c. \tag{12}$$

$$\mathbf{S}(\mathbf{r}, z, t) \cdot \mathbf{n}_z = \frac{1}{2}\sum_{m=1}^{\infty} A_m(z)\mathbf{S}_m(\mathbf{r}) \cdot \mathbf{n}_z e^{-i\omega t} \quad + \quad c.c. \tag{13}$$

where $\mathbf{v}_m = \partial u_m/\partial t$ is the particle velocity of the m^{th} mode, $\mathbf{S}_m$ is the stress tensor for the m^{th} mode, $\mathbf{n}_z$ is the unit vector in the wave propagation direction z, and A_m is the higher order modal amplitude to be determined. Notice that the 1/2 and complex conjugates are needed to ensure real quantities (de Lima and Hamilton [2]). As shown by Auld, A_m is the solution to the following ordinary differential equation, to be solved for each individual value of m:

$$4P_{mn}(\frac{d}{dz} - i\kappa_n^*)A_m(z) = (f_n^{\mathrm{surf}} + f_n^{\mathrm{vol}})e^{i\kappa z}\ ;\ m = 1, 2, ... \tag{14}$$

where,

$$P_{mn} = -\frac{1}{4}\int_{\Omega} (\mathbf{v_n^*} \cdot \mathbf{S_m} + \mathbf{v_m} \cdot \mathbf{S_n^*}) \cdot \mathbf{n}_z d\Omega, \tag{15}$$

$$f_n^{surf}(z) = \int_{\Gamma} \mathbf{v_n^*} \,\textbf{.}\, \bar{\mathbf{S}} \,\textbf{.}\, \mathbf{n_r} \, d\Gamma \tag{16}$$

$$f_n^{vol}(z) = \int_{\Omega} \mathbf{v_n^*} \,\textbf{.}\, \bar{\mathbf{f}} \, d\Omega \tag{17}$$

and κ_n is the wavenumber of the wave that is not orthogonal to the mode with wavenumber κ_m. $\bar{\mathbf{S}}$ and $\bar{\mathbf{f}}$ are the surface traction and body force, respectively, as given by the primary wave. Ω, Γ are the rod cross-sectional area and the rod surface, respectively (Fig. 1).

3.2 Method of Perturbation

De Lima and Hamilton [4] provided the solution to Eq. (14) for the case of first order nonlinearity. By using perturbation, the solution to Eqs. (1) and (2) is written as a sum of two components:

$$\mathbf{u} = \mathbf{u^1} + \mathbf{u^2} \tag{18}$$

where $\mathbf{u^2}$ is the perturbation due to nonlinearity and is assumed to be small in comparison to $\mathbf{u^1}$. $\mathbf{u^1}$ is the solution to the following linear problem:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u^1}) - \mu\nabla \times (\nabla \times \mathbf{u^1}) - \rho_0 \frac{\partial^2 \mathbf{u^1}}{\partial t^2} = 0, \tag{19}$$

$$\mathbf{S^L}(\mathbf{u^1}) \cdot \mathbf{n_r} = \mathbf{0} \quad \text{on} \quad \Gamma \tag{20}$$

which represents the solution to the classical linear rod problem with stress free boundary conditions. $\mathbf{u^2}$ is the solution to the forced problem:

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u^2}) - \mu\nabla \times (\nabla \times \mathbf{u^2}) - \rho_0 \frac{\partial^2 \mathbf{u^2}}{\partial t^2} = -\mathbf{f^1}, \tag{21}$$

$$\mathbf{S^L}(\mathbf{u^2}) \cdot \mathbf{n_r} = -\mathbf{S^1} \cdot \mathbf{n_r} \quad \text{on} \quad \Gamma \tag{22}$$

where $\mathbf{S^1}$ and $\mathbf{f^1}$ are surface traction and body force as calculated from the primary solution $\mathbf{u^1}$.

It must be noted that $\mathbf{S^1}$ and $\mathbf{f^1}$ are calculated by substituting the primary excitation into stress and force equations (10, 11). If the primary excitation is a guided wave mode at a frequency ω and the energy equation (3) is nonlinear to the first order, both $\mathbf{S^1}$ and $\mathbf{f^1}$ would consequently be harmonic at 2ω. Similarily, if the energy equation also contains 2^{nd} order nonlinearity, $\mathbf{S^1}$ and $\mathbf{f^1}$ would contain triple harmonic (3ω) terms.

The first-order nonlinear solution to Eq. (14) is:

$$A_m(z) = \bar{A}_m(z)e^{i(2\kappa z)} - \bar{A}_m(0)e^{i\kappa_n^* z}, \tag{23}$$

where,

$$\bar{A}_m(z) = i\frac{(f_n^{\mathrm{vol}} + f_n^{\mathrm{surf}})}{4P_{mn}[\kappa_n^* - 2\kappa]} \; ; \; \kappa_n^* \neq 2\kappa \quad (asynchronous\ solution) \tag{24}$$

$$\bar{A}_m(z) = \frac{(f_n^{\mathrm{vol}} + f_n^{\mathrm{surf}})}{4P_{mn}} z \; ; \; \kappa_n^* = 2\kappa \quad (synchronous\ solution) \tag{25}$$

where A_m are the amplitudes of the modes at 2ω and κ is the wavenumber of the primary wave.

4 Analysis of solution

The first order nonlinear solution (Eq. 23) can be extended to higher orders by using appropriate $\bar{\mathbf{S}}$ and $\bar{\mathbf{f}}$ in Eqs. (16) and (17) and by using normal mode expansion at the appropriate higher harmonic. Since the method of perturbation reduces the nonlinear problem to a forced linear problem, it can, for the sake of simplicity, be assumed that the energy expression, (Eq. (3)) consists of any one single order of nonlinearity. Consequently, $\bar{\mathbf{S}}$ and $\bar{\mathbf{f}}$ are due to that particular order of nonlinearity alone. For an $(n-1)^{\mathrm{th}}$ order nonlinearity, these are denoted by $\bar{\mathbf{S}}^{\mathbf{n}}$ and $\bar{\mathbf{f}}^{\mathbf{n}}$. The subscripts for f^{surf} and f^{vol} are, therefore, changed to 'l' for the sake of clarity.

For the specific case of a cylindrical rod of radius a, Eqs. (16) and (17) become:

$$f_l^{surf} = -\frac{a}{2}\int_0^{2\pi} \mathbf{v}_l^*(a,\theta) \,.\, \bar{\mathbf{S}}^{\mathbf{n}}(a,\theta) \,.\, \mathbf{n}_{\mathbf{r}} \, d\theta \tag{26}$$

$$f_l^{vol} = \frac{1}{2}\int_0^{a}\int_0^{2\pi} \mathbf{v}_l^*(r,\theta) \,.\, \bar{\mathbf{f}}^{\mathbf{n}}(r,\theta) r \, d\theta \, dr \tag{27}$$

where the superscript n refers to the primary excitation mode at the $n\omega$ frequency, and the subscript l refers to the potential higher-harmonic generation. The particle velocity for the l^{th} Pochhammer Chree wave in rods at frequency $n\omega$ is (Meitzler [7]):

$$v_r = V_r(r)\cos(q\theta)e^{i(\kappa_l z - n\omega t)} \tag{28}$$

$$v_\theta = V_\theta(r)\sin(q\theta)e^{i(\kappa_l z - n\omega t)} \tag{29}$$

$$v_z = V_z(r)\cos(q\theta)e^{i(\kappa_l z - n\omega t)} \tag{30}$$

where q is an integer related to the family of modes. $q = 0$ for longitudinal modes, and $q \geq 1$ for flexural modes. Torsional modes do not depend upon θ and will not be dealt with here. Substituting the expressions for f_l^{surf} and f_l^{vol} and ignoring the exponential harmonic term yields:

$$f_l^{surf} = -\frac{a}{2}\int_0^{2\pi}\left[V_r\bar{S}_{rr}^n\cos(q\theta) + V_\theta\bar{S}_{\theta r}^n\sin(q\theta) + V_z\bar{S}_{zr}^n\cos(q\theta)\right]d\theta \tag{31}$$

$$f_l^{vol} = \frac{1}{2}\int_0^a\int_0^{2\pi}[V_r\bar{f}_r^n\cos(q\theta) + V_\theta\bar{f}_\theta^n\sin(q\theta) + V_z\bar{f}_z^n\cos(q\theta)]r d\theta\, dr \tag{32}$$

The following identities must be noted for all integers $n \neq 0$:

$$\int_0^{2\pi}\sin(n\theta)\,d\theta = 0 \tag{33}$$

$$\int_0^{2\pi} \cos(n\theta)\ d\theta = 0 \tag{34}$$

From Eqs. (24) and (25), the l^{th} mode would not be excited if both $f_l^{surf}, f_l^{vol} = 0$. This is possible if all the terms in Eqs. (31) and (32) are identically equal to zero. Hence if all the terms in Eqs. (31) and (32) can be expressed as one of the two integrals in Eqs. (33) and (34) for an l^{th} mode, that particular mode could not be excited at the higher harmonic.

At this point, the following trigonometric identities should also be noted. These can be deduced using De Moivre's formula, Euler's formula and binomial expansion:

if n is odd:

$$\cos^n(\theta) = \sum_{k=0}^{\frac{n-1}{2}} A_k \cos(n-2k)\theta \tag{35}$$

$$\sin^n(\theta) = \sum_{k=0}^{\frac{n-1}{2}} B_k \sin(n-2k)\theta \tag{36}$$

where A_k and B_k are only functions of n.

If n is even:

$$\cos^n(\theta) = X + \sum_{k=0}^{\frac{n}{2}-1} C_k \cos(n-2k)\theta \tag{37}$$

$$\sin^n(\theta) = X + \sum_{k=0}^{\frac{n}{2}-1} D_k \cos(n-2k)\theta \tag{38}$$

where X, C_k and D_k are only functions of n.

For an $(n-1)^{\text{th}}$ order nonlinearity the energy expression Eq. (3) contains terms which have $(n+1)$ multiples of strains. Therefore from the velocity expressions, Eqs. (28-30), the corresponding nonlinear stress tensor $\bar{\mathbf{S}}^{\mathbf{n}}$ and body force vector $\bar{\mathbf{f}}^{\mathbf{n}}$ contain n multiples of strains, Eqs. (10, 11).

Hence for a $(n-1)^{\text{th}}$ order nonlinearity, any generic term in the stress $(\bar{\mathbf{S}}^{\mathbf{n}})$ and body force $(\bar{\mathbf{f}}^{\mathbf{n}})$ can be expressed as:

$$T^n = f(r)\sin^t(p\theta)\cos^s(p\theta) \qquad t+s=n \tag{39}$$

where p is related to the family of primary excitation mode, $f(r)$ is an arbitrary function of the radius r and either t or s can be equal to 0.

From Eqs. (31), (32) it can be seen that each term in the expressions for f_l^{surf} and f_l^{vol} involves an integral of the form:

$$I^n = \int_0^{2\pi} F(r)\sin^t(p\theta)\cos^s(p\theta)\sin(l\theta)\,d\theta \tag{40}$$

or

$$I^n = \int_0^{2\pi} F(r)\sin^t(p\theta)\cos^s(p\theta)\cos(l\theta)\,d\theta \tag{41}$$

For the ease of analysis, we denote $\sin^t(p\theta)\cos^s(p\theta) = E^n$.

4.1 Case 1: n is odd (odd harmonics)

Since n is odd and $t+s=n$, either t is odd or s is odd. Assuming that t is odd, we have the following expansion:

$$E^n = \tag{42}$$

$$\left(\sum_{k_1=0}^{\frac{t-1}{2}} B_{k_1} \sin\{(t-2k_1)p\theta\}\right)\left(X+\sum_{k_2=0}^{\frac{s}{2}-1} C_{k_2}\cos\{(s-2k_2)p\theta\}\right)$$

$$= (X\sum_{k_1=0}^{\frac{t-1}{2}} B_{k_1}\sin\{(t-2k_1)p\theta\} +$$

$$\sum_{k_1=0}^{\frac{t-1}{2}}\sum_{k_2=0}^{\frac{s}{2}-1}(B_{k_1}\sin\{(t-2k_1)p\theta\}\ C_{k_2}\cos\{(s-2k_2)p\theta\}) \tag{43}$$

Further

$$S =$$

$$\sum_{k_1=0}^{\frac{t-1}{2}}\sum_{k_2=0}^{\frac{s}{2}-1} B_{k_1}\sin\{(t-2k_1)p\theta\}\ C_{k_2}\cos\{(s-2k_2)p\theta\}$$

$$= \sum_{k_1=0}^{\frac{t-1}{2}}\sum_{k_2=0}^{\frac{s}{2}-1}\frac{1}{2}B_{k_1}C_{k_2}\left(\sin\{(t+s-2k_1-2k_2)p\theta\}+\sin\{(t-2k_1-s+2k_2)p\theta\}\right)$$

Now the term $(t+s-2k_1-2k_2)$ assumes all odd numbers between 3 ($k_1=\frac{t-1}{2}$, $k_2=\frac{s}{2}-1$) and $t+s$ ($k_1=k_2=0$). Similarly, $(t-2k_1-s+2k_2)$ assumes only odd values. It assumes a value of -1 ($k_1=\frac{t-1}{2}$, $k_2=\frac{s}{2}-1$) which is equivalent to 1 if the negative sign is taken out of the sine term. In other words, S can be expressed as:

$$S = \sum_{k=1,3\ldots}^{t+s} B_k \sin(kp\theta) \tag{44}$$

where B_k are constants. Substituting this in Eq. (42), after some algebraic manipulations:

$$E^n = \tag{45}$$
$$\sum_{k_1=1,3,\ldots}^{t} B_{k_1}\sin(k_1 p\theta) + \sum_{k_2=1,3\ldots}^{t+s} B_{k_2}\sin(k_2 p\theta) = \sum_{k=1,3,\ldots}^{t+s} E_k \sin(kp\theta)$$

Similarly, it can be shown that if t is even, we have the following expansions:

$$E^n = \tag{46}$$
$$\left(X + \sum_{k_1=0}^{\frac{t}{2}-1} D_{k_1}\cos\left\{(t-2k_1)p\theta\right\}\right)\left(\sum_{k_2=0}^{\frac{s-1}{2}} A_{k_2}\cos\left\{(t-2k_2)p\theta\right\}\right)$$
$$= \sum_{k=1,3\ldots}^{t+s} E_k \cos(kp\theta)$$

If Eq. (40) holds, we have:

$$I^n = \int_0^{2\pi} F(r)\left(\sum_{k=1,3\ldots}^{t+s} E_k \sin(kp\theta)\right)\sin(l\theta)\,d\theta \quad ; \quad t\ odd \tag{47}$$

or

$$I^n = \int_0^{2\pi} F(r)\left(\sum_{k=1,3\ldots}^{t+s} E_k \cos(kp\theta)\right)\sin(l\theta)\,d\theta \quad ; \quad t\ even \tag{48}$$

The integral in Eq. (48) is always 0 whereas the integral in Eq. (47) is nonzero if and only if $l = kp$ for some value of k.

On the other hand if Eq. (41) holds, we have:

$$I^n = \int_0^{2\pi} F(r)\left(\sum_{k=1,3\ldots}^{t+s} E_k \sin(kp\theta)\right)\cos(l\theta)\,d\theta \quad ; \quad t\ odd \tag{49}$$

or

$$I^n = \int_0^{2\pi} F(r) \left(\sum_{k=1,3...}^{t+s} E_k \cos(kp\theta) \right) \cos(l\theta)\, d\theta \quad ; \quad t \textit{ even} \tag{50}$$

The integral in Eq. (49) is always 0 whereas the integral in Eq. (50) is nonzero if and only if $l = kp$ for some value of k. Hence, I_n in all the above cases is nonzero, iff $l = kp$ for some value of k.

Therefore a primary flexural mode (designated by an angular order $p \neq 0$) cannot generate a longitudinal mode ($l = 0$) at an odd higher harmonic (n odd). Vice-versa, a primary longitudinal mode ($p = 0$) cannot generate a flexural mode ($l \neq 0$) at an odd higher harmonic (n odd). Further, a primary flexural mode ($p \neq 0$) can only generate at the n^{th} harmonic, those modes for which $l = kp$ where $k = 1, 3...n$. No higher order modes can be generated at that harmonic.

4.2 Case 2: n is even (even harmonics)

Since n is even and $t + s = n$, either both t and s are odd or both are even. Assuming that both t and s are odd, we have the following expansion:

$$\begin{aligned} E^n = & \\ & \left(\sum_{k_1=0}^{\frac{t-1}{2}} B_{k_1} \sin\{(t-2k_1)p\theta\} \right) \left(\sum_{k_2=0}^{\frac{s-1}{2}} A_{k_2} \cos\{(s-2k_2)p\theta\} \right) \\ & = \sum_{k_1=0}^{\frac{t-1}{2}} \sum_{k_2=0}^{\frac{s-1}{2}} B_{k_1} \sin\{(t-2k_1)p\theta\}\ A_{k_2} \cos\{(s-2k_2)p\theta\}) \end{aligned} \tag{51}$$

It can be shown that the above expression reduces to:

$$E^n = \tag{52}$$
$$\sum_{k=2,4,\ldots}^{t+s} E_k \sin(kp\theta) = \sum_{k=0,2,\ldots}^{t+s} E_k \sin(kp\theta)$$

Similarly, when t and s are even, we have:

$$E^n = \tag{53}$$
$$\left(X + \sum_{k_1=0}^{\frac{t}{2}-1} D_{k_1} \cos\left\{(t-2k_1)p\theta\right\} \right) \left(X + \sum_{k_2=0}^{\frac{s}{2}-1} C_{k_2} \cos\left\{(s-2k_2)p\theta\right\} \right)$$
$$= Y + \sum_{k=2,4,\ldots}^{t+s} E_k \cos(kp\theta) = \sum_{k=0,2,\ldots}^{t+s} E_k \cos(kp\theta)$$

where Y contains all terms independent of θ. If Eq. (40) holds, we have:

$$I^n = \int_0^{2\pi} F(r) \left(\sum_{k=0,2\ldots}^{t+s} E_k \sin(kp\theta) \right) \sin(l\theta)\, d\theta \quad ; \quad t, s \; odd \tag{54}$$

or

$$I^n = \int_0^{2\pi} F(r) \left(\sum_{k=0,2\ldots}^{t+s} E_k \cos(kp\theta) \right) \sin(l\theta)\, d\theta \quad ; \quad t, s \; even \tag{55}$$

The integral in Eq. (55) is always 0, whereas the integral in Eq. (54) is nonzero if and only if $l = kp$ for some value of $k \neq 0$.

On the other hand if Eq. (41) holds, we have:

$$I^n = \int_0^{2\pi} F(r) \left(\sum_{k=0,2\ldots}^{t+s} E_k \sin(kp\theta) \right) \cos(l\theta)\, d\theta \quad ; \quad t, s \; odd \tag{56}$$

or

$$I^n = \int_0^{2\pi} F(r) \left(\sum_{k=0,2...}^{t+s} E_k \cos(kp\theta) \right) \cos(l\theta)\, d\theta \quad ; \quad t, s\ even \tag{57}$$

The integral in Eq. (56) is always 0, whereas the integral in Eq. (57) is nonzero iff $l = kp$ for some value of k. Since f^{surf} and f^{vol} in Eqs. (31) and (32) are in general a combination of the above terms, it can be said that a primary mode of order p can generate a higher order mode with an angular order l iff $l = kp$ for some value of $k = 0, 2...n$.

Therefore even a primary flexural mode (designated by an angular order $p \neq 0$) can generate a longitudinal mode ($l = 0$ for $k = 0$) at an even higher harmonic (n even). Vice-versa, a primary longitudinal mode ($p = 0$) cannot still generate a flexural mode ($l \neq 0$) at an even higher harmonic (n even).

5 Conclusions

It can be seen from the above analysis that a primary generating mode in a rod with an angular order p will generate a rod mode with an angular order l at the n^{th} higher harmonic if and only if $l = kp$ for some values of k where:

1. k spans all odd numbers from 1 to n when n is odd (odd harmonics).

2. k spans all even numbers from 0 to n when n is even (even harmonics).

Therefore, the conclusions applicable to *odd* harmonics are:

1. A longitudinal primary generating mode will not produce any flexural modes. A longitudinal primary generating mode can only produce longitudinal modes.

2. Only selected modes can be generated by flexural primary generating modes. For example, a first order flexural mode ($p = 1$) at the triple harmonic ($n = 3$) can only generate the first order ($l = 1$) and third order ($l = 3$) flexural modes. In general, a p^{th} order flexural mode can only generate at the n^{th} harmonic flexural modes of orders equal to odd multiples of p, up to np.

The conclusions applicable to *even* harmonics are:

1. Longitudinal modes can be generated irrespective of whether the primary generating mode is longitudinal or flexural. Moreover, a longitudinal primary generating mode does not produce flexural modes.

2. As in the case of odd harmonics, only selected modes can be generated by flexural primary generating modes. For example, a first order flexural mode ($p = 1$) at the double harmonic can only generate the longitudinal ($l = 0$) and second order ($l = 2$) flexural modes. In general, a p^{th} order flexural mode can only generate at the n^{th} harmonic the longitudinal mode and flexural modes of orders equal to even multiples of p, up to np.

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List of figure captions

fig1: Schematic of a stress free rod.